

# Bohmian Mechanics at Space-Time Singularities.

## I. Timelike Singularities

Roderich Tumulka\*

August 1, 2007

### Abstract

We develop an extension of Bohmian mechanics by defining Bohm-like trajectories for (one or more) quantum particles in a curved background space-time containing a singularity. Part one, the present paper, focuses on timelike singularities, part two will be devoted to spacelike singularities. We use the timelike singularity of the (super-critical) Reissner–Nordström geometry as an example. While one could impose boundary conditions at the singularity that would prevent the particles from falling into the singularity, in the case we are interested in here particles have positive probability to hit the singularity and get annihilated. The wish for reversibility, equivariance and the Markov property then dictate that particles must also be created by the singularity, and indeed dictate the rate at which this must occur. That is, a stochastic law prescribes what comes out of the singularity. We specify explicit model equations, involving a boundary condition on the wave function at the singularity, which is applicable also to other versions of quantum theory besides Bohmian mechanics.

Key words: quantum theory in curved background space-time; Reissner–Nordstrom space-time geometry; timelike singularities; Bohmian trajectories; particle creation and annihilation; stochastic jump process.

## 1 Introduction

We study the construction and behavior of quantum mechanics in a space-time with (timelike) singularities, using Bohmian mechanics (also known as pilot-wave theory), a version of quantum mechanics with particle world lines. We develop an explicit model, defining trajectories on a given background space-time geometry, using the Dirac equation, Bohm’s law of motion, and a novel boundary condition at the singularity that governs the creation and annihilation of particles at the singularity. As a case study,

---

\*Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA. E-mail: tumulka@math.rutgers.edu

we consider the Reissner–Nordström geometry in the charge  $>$  mass regime, featuring a naked timelike singularity.

Bohmian mechanics was developed as a realist version of nonrelativistic quantum mechanics [4] and succeeds in explaining all phenomena of quantum mechanics in terms of an objective, observer-independent reality consisting of point particles moving in space; see [13] for an overview. Bohmian mechanics possesses a natural extension to relativistic space-time if a preferred foliation of space-time into spacelike hypersurfaces (called the *time foliation*) is granted [7]. This extension has also been formulated for curved space-time geometries [25, 26], but not yet for geometries with singularities. While horizons present no difficulty, singularities require further work to define the theory: Basically, we have to specify what happens when a particle hits the singularity, since at this point the law of motion is no longer defined. The possibility we consider here is that the particle gets annihilated: that is, if the system consisted of  $N$  particles, then the world line of the particle that has arrived at the singularity ends there, while the other  $N - 1$  particles, which are not at the singularity and thus have no reason to vanish, continue to move according to Bohm’s law of motion. To make this possible, we need wave functions from Fock space, as always when particles can get created or annihilated. Further considerations then naturally lead us to specific equations, defining a Bohm-type theory.

## 1.1 Particle Creation

The most astonishing feature of this theory is perhaps that particles can also be *created* at the singularity. Indeed, this occurs at random times distributed in a way determined by a law involving the wave function. As a consequence, the theory is no longer deterministic; instead, the evolution of the particles can be described by a Markovian stochastic process in the appropriate configuration space. This process has similarities with processes used earlier in “Bell-type quantum field theories” [10], a stochastic version of Bohmian mechanics for quantum field theories that we describe briefly in Section 6.

It is an interesting moral from our model that particle creation and annihilation, one of the characteristic features of quantum field theory, can arise from the presence of timelike singularities. This invites speculation whether quantum particles capable of emitting other particles might actually contain timelike singularities. After all, the classical gravitational field of a point particle with the charge and the mass of an electron is exactly the Reissner–Nordström geometry we are considering. This does not necessarily mean that the ultimate theory of the electron, including quantum gravity, will attribute a timelike singularity to it, but at least this seems like an intriguing possibility.

Another moral from our model is that timelike singularities are not as absurd as one might conclude from classical physics. Quantum mechanics (as well as Bohmian mechanics) differs in this respect from classical mechanics, which breaks down at timelike singularities because *anything* could come out of a timelike singularity, and there is no good reason for postulating any law about when and where particle world lines begin at the singularity. This is different in Bohmian mechanics because, as soon as we have defined the evolution of a wave function in Fock space, the probability current given in

terms of the wave function, such as

$$j^\mu = \bar{\psi} \gamma^\mu \psi, \quad (1)$$

defines the probability of particle creation at the singularity. Indeed, the stochastic law for particle creation is uniquely fixed if we strive for the following properties: reversibility, equivariance, and the Markov property. Thus, the Bohmian particle positions are governed by two laws: an equation of motion, and a law prescribing the stochastic creation of new particles. Both Bohmian and classical mechanics become *indeterministic* in the presence of timelike singularities, but in different ways: while Bohmian mechanics becomes *stochastic*, classical mechanics becomes *lawless*. One could say that our stochastic law closes exactly the gap in the laws caused by the timelike singularity. A timelike singularity causes concern in classical mechanics, but not in quantum mechanics.

## 1.2 Horizons

While the Reissner–Nordström example does not contain a horizon, we have a few remarks about the status of horizons in Bohmian mechanics.

Consider a many-particles quantum system in a background space-time containing a black hole. From the point of view of orthodox quantum mechanics, it is natural to trace out all degrees of freedom inside the event horizon. From the Bohmian viewpoint, in contrast, this is not natural. Instead, it is natural to ask what happens behind the horizon. This is so because from the orthodox viewpoint, the most important question is what an observer will see, and it is a frequent assumption that the relevant observers sit at infinity. From the Bohmian viewpoint, the most important question is what actually happens. Thus, to define Bohmian mechanics in a curved space-time, we need to define as well the trajectories inside the black hole. (A viewpoint that, like the “Copenhagen” view of quantum mechanics, dismisses any theory of particle positions because it regards the latter as “hidden variables,” may naturally tend to dismiss as unreal also everything that is hidden behind a horizon.)

Thus, the Bohmian viewpoint leads us to the following attitude: What happens inside a black hole *can and should* be described by a physical theory.

Indeed, by the nonlocal character of Bohmian mechanics, the velocities of the particles outside the black hole may depend on the positions of the particles inside the black hole. But all this requires no additional research for the definition of the theory, as the Bohm-type law of motion we use (see Eq. (4) below) automatically implies influences across event horizons. What requires further work is not the presence of a horizon, but the presence of a singularity. (The need for this further work would disappear if none of the 3-surfaces belonging to the time foliation touched the singularity. However, for our purposes that is the uninteresting case. Furthermore, I see no reason why the time foliation should avoid the singularity.)

### 1.3 Concrete Model

The concrete model we develop is based on the Dirac equation and the Reissner–Nordström geometry, but we expect that similar models can be constructed on the basis of other wave equations (such as the Weyl equation for massless right-handed spin- $\frac{1}{2}$  particles, or equations for higher spins) and other space-times containing timelike singularities. We postpone the discussion of spacelike singularities to a future work; lightlike singularities behave, for our purposes, like spacelike ones. (A general singularity may well, like a general 3-surface, have different regions in which it is timelike, spacelike, and lightlike. Correspondingly, our separate studies of timelike and spacelike singularities apply separately to the timelike and the spacelike subset of the singularity.)

The Dirac equation is known to have solutions of negative energy that are considered unphysical; indeed, the Hamiltonian is unbounded from below. We will not worry about this here, but rather treat the negative energy states as if they were physical. However, I recognize that, in a further development of this theory, the evolution should be so defined that only those states normally considered as physical can arise (in return, it may involve pair creation). A difficulty one encounters when trying to write down such a model is the question: Which subspace exactly of the (one-particle) Hilbert space should be regarded as the space of physical states? Since the gap of  $2mc^2$  in the spectrum of the Dirac Hamiltonian may disappear in the presence of an external gravitational (or, for that matter, electromagnetic) field, there is no longer any natural, gauge invariant, Lorentz invariant way of splitting Hilbert space into positive and negative energies. So we leave this question open.

We also have to leave open the question whether the Hamiltonian we suggest here for the evolution of  $\psi$  is truly self-adjoint; but there are reasons to believe this. If this proves right then our model will be an example of a theory of particle creation without an ultraviolet cut-off. In other words, in this setting the ultraviolet divergence which usually plagues any quantum theory with particle creation and annihilation will be absent, in fact without any special procedure à la renormalization. This encourages hope that the extreme curvature near a point mass may actually resolve, or at least ameliorate, the ultraviolet problem of QED and other quantum field theories.

### 1.4 Motivation

The investigations in this paper have several motivations, and lead to gains, in several areas:

- It is a natural part of the research program on Bohmian mechanics to extend the theory to more general quantum theories, to all kinds of settings. To the extent that we have reason to believe that singularities exist in our universe, we obtain here a more appropriate version of Bohmian mechanics. Concerning techniques of constructing Bohm-type models, we find that the Bohm-type law of motion proposed by Dürr *et al.* [7] for relativistic space-time with a foliation works unproblematically even under the extreme conditions near a singularity; that not

every spacelike foliation is equally good for that purpose (as we discuss in Section 7); that the stochastic approach to particle creation developed in “Bell-type quantum field theories” [10] arises naturally in this context; that particle annihilation may well be deterministic and particle creation stochastic, even though the evolution is reversible.

- Since Bohmian mechanics is a particularly precise and unambiguous version of quantum mechanics it may serve as a tool for studying quantum mechanics in curved space-time. Thus, our study can as well be regarded as one on *quantum mechanics at space-time singularities*. Concretely, we obtain a novel unitary evolution of the wave function in the presence of a timelike singularity, based on the Dirac equation and a boundary condition at the singularity (independently of the Bohmian viewpoint).
- Several interesting features arise in the theory we develop: Timelike singularities emit particles; we can specify the law for how they do it; certain traits of quantum field theory (such as Fock space) arise from the presence of singularities; timelike singularities need not be unphysical. Finally, an ultraviolet regularization may not be necessary, thus suggesting a way out of the ultraviolet divergence for quantum field theories.

I emphasize that the theory developed in this paper does not merely add Bohmian trajectories to otherwise known “orthodox” quantum theories. Instead, the Bohmian viewpoint helps define what the evolution equation should be, also for the wave function. Indeed, the Fock space evolution corresponding to particle creation at a timelike singularity *could be, but has not been*, considered in an orthodox framework. The reason why it has not been considered before is, I think, twofold: First, since orthodox quantum physics tends not to ask what actually happens but rather what observers see, it is common to trace out the degrees of freedom inside a black hole, which means to ignore the phenomena this paper is concerned with. Second, in orthodox quantum physics it is common to focus on the scattering regime, i.e., on the asymptotic distribution of particles a long time after their interaction. For defining a Bohm-like theory, in contrast, it is essential to define the trajectories, which may well depend on the particles hidden behind the event horizon, and this forces us to pay attention to what the time evolution laws are. Note also that a scattering theory can only be formulated in an asymptotically flat space-time, while our approach does not require asymptotic flatness or stationarity, even though the Reissner–Nordström geometry happens to be asymptotically flat and stationary.

The paper is organized as follows. In Section 2 we recall the relevant version of Bohm’s law of motion. In Section 3 we recall the Reissner–Nordström geometry. In Section 4 we discuss the general framework of particle creation and annihilation at a timelike singularity. In Section 5 we describe the concrete model. In Section 6 we draw parallels between our model and Bell-type quantum field theories. In Section 7 we describe a property of the time foliation that we have used.

## 2 Bohmian Mechanics

### 2.1 Why Bohmian Mechanics?

Bohmian mechanics is one of the most promising candidates for how quantum phenomena work. While the quantum formalism describes what observers will see, and while the conventional “Copenhagen” view of quantum mechanics remains vague and paradoxical, Bohmian mechanics provides a possible explanation of the quantum formalism in terms of objective events, in fact by postulating that particles have actual positions and hence trajectories. Bohmian mechanics is well understood in the realm of non-relativistic quantum mechanics, but needs further development in the directions of relativistic physics, quantum field theory, and quantum gravity. This paper concerns the relativistic extension in a classical gravitational field, but connects also with quantum field theory.

The nature of quantum reality remains controversial. The traditional Copenhagen view is vague about what is real and what is not, and leaves the process of observation unanalyzed. To be sure, the predictions for the possible results of quantum experiments and their probabilities are uncontroversial. However, we are dissatisfied with the Copenhagen view and feel that the quantum mechanical predictions call for an *explanation*. A good theory should provide an account of what objectively happens. Whether this account is *true* may be hard or impossible to decide, but we should require that it be *consistent*, in itself and with the quantum mechanical probabilities, while avoiding the vagueness of the Copenhagen version of quantum mechanics.

Such an account is provided by Bohmian mechanics [4, 3, 5, 11]; see [13] for an overview. This theory postulates that particles have trajectories, governed by an equation of motion of the type

$$\frac{dQ_t}{dt} = \frac{j^\psi(Q_t)}{\rho^\psi(Q_t)}, \quad (2)$$

where  $Q_t$  is the position of the particle at time  $t$  (or, for a system of several particles, the *configuration*), and  $j^\psi$  and  $\rho^\psi$  are, respectively, the quantum mechanical probability current and probability density as determined by the wave function  $\psi$ . For example, if  $\psi$  is a Dirac wave function then

$$j^\psi = \psi^\dagger \alpha \psi, \quad \rho^\psi = \psi^\dagger \psi. \quad (3)$$

As a consequence of the structure (2) of the law of motion, if at any time  $t$  the particle position (or configuration) is random with distribution  $\rho^{\psi_t}$ , then this is also true of any other time  $t$ . This property is called *equivariance*. As a (quite non-obvious) consequence of *that*, inhabitants of a Bohmian universe, consisting of these particles with trajectories, would observe the same probabilities in their experiments as predicted by the quantum formalism [11]. That is how Bohmian mechanics explains quantum mechanics. In fact, Bohmian mechanics accounts for all phenomena of non-relativistic quantum mechanics.

It is one of the virtues of the Bohmian viewpoint that it forces us to face those difficulties that can easily be swept under the rug in the orthodox framework.

## 2.2 Bohmian Mechanics and Relativity

With the invocation of a preferred foliation  $\mathcal{F}$  of space-time into spacelike 3-surfaces, given by a Lorentz invariant law and called the *time foliation*, it is known [5, 17, 7, 25] that Bohmian mechanics possesses a natural generalization to relativistic space-time. The possibility of a preferred foliation seems against the spirit of relativity (see [17] for a discussion), but certainly worth exploring. It is suggested by the empirical fact of quantum non-locality, and it is suggested by the structure of the Bohmian law of motion (2) for many particles, in which the velocity of a particle depends on the instantaneous position of the other particles. (But see [14] for an indication of how a relativistic Bohm-like theory might be able to dispense with a preferred foliation.) Using a time foliation  $\mathcal{F}$ , a Bohm-type equation of motion was formulated in [7] for flat space-time (based on earlier work in [5]), and the straightforward generalization to curved space-time was formulated and mathematically studied in [25]:

$$\frac{dX_k^{\mu_k}}{d\tau} \propto j^{\mu_1 \dots \mu_N}(X_1(\Sigma), \dots, X_N(\Sigma)) \prod_{i \neq k} n_{\mu_i}(X_i(\Sigma)), \quad (4)$$

where  $X_k(\tau)$  is the world line of particle  $k \in \{1, \dots, N\}$ ,  $\tau$  is any curve parameter,  $\Sigma$  is the 3-surface in  $\mathcal{F}$  containing  $X_k(\tau)$ ,  $n(x)$  is the unit normal vector on  $\Sigma$  at  $x \in \Sigma$ ,  $X_i(\Sigma)$  is the point where the world line of particle  $i$  crosses  $\Sigma$ , and

$$j^{\mu_1 \dots \mu_N} = \bar{\psi}(\gamma^{\mu_1} \otimes \dots \otimes \gamma^{\mu_N})\psi \quad (5)$$

is the probability current associated with the  $N$ -particle Dirac wave function  $\psi$ . This wave function could either be a multi-time wave function defined on  $(\text{space-time})^N$ , or, since we never use  $\psi$  for configurations that are not simultaneous, it suffices that  $\psi$  is defined on the  $3N + 1$ -dimensional manifold  $\bigcup_{\Sigma \in \mathcal{F}} \Sigma^N$  of simultaneous configurations.

This generalization does not automatically include, however, space-time geometries with singularities. The treatment of singularities requires, as we point out in Section 4 below, some fundamental extensions of Bohmian mechanics, and forms a test case for the robustness of the equation of motion (4).

As mentioned before, the foliation might itself be dynamical. An example of a possible Lorentz invariant evolution law for the foliation is

$$\nabla_\mu n_\nu - \nabla_\nu n_\mu = 0, \quad (6)$$

which is equivalent to saying that the infinitesimal timelike distance between two nearby 3-surfaces from the foliation is constant along the 3-surface. This law allows to choose an initial spacelike 3-surface and then determines the foliation. A special foliation obeying (6) is the one consisting of the surfaces of constant timelike distance from the Big Bang. Note, however, that the law of motion (4) does not require any particular choice of law for the foliation, except that the foliation does not depend on the particle configuration (while it may depend on the wave function). Note further that in a space-time with horizons, a foliation law like (6) will frequently lead to 3-surfaces lying partly outside

and partly inside the horizon. It will turn out from our study of singularities that (6) is not a good law for the time foliation in the presence of a timelike singularity as the foliations obeying (6) cannot be extended onto the singularity; see Section 7 for details.

### 3 Reissner–Nordström Geometry

Our main example of a timelike singularity will be the one of the *Reissner–Nordström* space-time geometry [19, 22, 27, 20, 18, 15], a solution of the coupled Einstein and Maxwell equations for a charged point mass. It is given by

$$ds^2 = -\lambda(r)dt^2 + \frac{1}{\lambda(r)}dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \quad (7)$$

with

$$\lambda(r) = 1 - \frac{2M}{r} + \frac{e^2}{r^2} \quad (8)$$

and parameters  $M > 0$  and  $e \in \mathbb{R}$ . The Reissner–Nordström metric is spherically symmetric, static, asymptotically flat, and singular at  $r = 0$ . It arises as the gravitational field of a (non-rotating) point particle with mass  $M$  and charge  $e$ . We assume the so-called *super-extremal* case  $|e| > M$ ; in this case  $\lambda(r) > 0$  for all  $r \geq 0$ . In this context we note that if an electron were a classical point particle, its gravitational field would be a super-extremal Reissner–Nordström field containing a timelike singularity. This does not mean that electrons must, but does suggest they might, contain timelike singularities in nature.

We will find it more useful to regard the singularity  $\{r = 0\}$  not as a *line* (as suggested by the picture of a point particle, and by the fact that in Minkowski space-time  $\{r = 0\}$  is a line) but as a *surface* (of topology  $\mathbb{R} \times \mathbb{S}^2$ ). Thus, the space-time

$$\mathcal{M} \cong \{(t, r, \omega) : t \in \mathbb{R}, r \geq 0, \omega \in \mathbb{S}^2\} = \mathbb{R} \times [0, \infty) \times \mathbb{S}^2 \quad (9)$$

(where  $\cong$  means diffeomorphic) is a manifold with boundary. (For the mathematical notion of a manifold with boundary, see [16, 1]. We will sometimes write  $\omega$  for a point on  $\mathbb{S}^2$ , and sometimes use the standard parameterization  $(\vartheta, \varphi)$  by polar and azimuthal angle.) The singularity is the *boundary*

$$S = \{r = 0\} = \partial\mathcal{M} \cong \mathbb{R} \times \{0\} \times \mathbb{S}^2, \quad (10)$$

while the metric  $g_{\mu\nu}$  is defined on the *interior*  $\mathcal{M}^\circ \cong \mathbb{R} \times (0, \infty) \times \mathbb{S}^2$ .

As the *time foliation*  $\mathcal{F}$  for defining the Bohmian trajectories we use the foliation provided by the time coordinate  $t$ , i.e., the family of level surfaces of the  $t$  function:

$$\mathcal{F} = \{\Sigma_s : s \in \mathbb{R}\} \quad \text{with} \quad \Sigma_s = \{x \in \mathcal{M} : t(x) = s\}. \quad (11)$$

Concretely,  $\Sigma_s = \{s\} \times [0, \infty) \times \mathbb{S}^2$ . The fact that we are considering a static space-time invites us to regard  $\Sigma := [0, \infty) \times \mathbb{S}^2$  as “the” 3-space.



## 4 Causal Trajectories at a Timelike Singularity

Bohmian trajectories, the integral curves of the probability current, are *causal* curves, i.e., their tangent vectors are always timelike or lightlike (when based on the Dirac equation). Such a curve may well reach the singularity  $\partial\mathcal{M}$  (indeed, in finite coordinate time  $t$  and finite proper time). What should happen if we consider a system of  $N$  particles and one of them hits the singularity? The simplest possibility, or at least a natural possibility is that this particle gets annihilated, that it stops existing, and history proceeds with  $N - 1$  particle. This is the possibility we will study in this paper.

In Bohmian theories, a configuration  $q$  of  $N$  identical particle is described by a set of  $N$  points in 3-space; thus, if  $\Sigma$  is 3-space, the configuration space is

$$\mathcal{Q}_N = \{q \subseteq \Sigma : \#q = N\}. \quad (12)$$

The wave function determining the motion of  $N$  particles is best regarded as a function on  $\widehat{\mathcal{Q}}_N$ , which can be taken to be the covering space of  $\mathcal{Q}_N$ , or, somewhat simpler,  $\widehat{\mathcal{Q}}_N = \Sigma^N$ . If the number of particles can change then we need several wave functions, one for each particle number. This leads us to considering state vectors in *Fock space*

$$\mathcal{H} = \bigoplus_{N=0}^{\infty} \mathcal{H}_N, \quad (13)$$

where  $\mathcal{H}_N$  is the Hilbert space of  $N$ -particle wave functions,  $\mathcal{H}_N = S_{\pm} \mathcal{H}_1^{\otimes N}$ , where  $S_+$  means the symmetrizer (appropriate for bosons) and  $S_-$  the anti-symmetrizer (for fermions). Every element  $\psi \in \mathcal{H}$  can thus be regarded as a sequence

$$\psi = (\psi_0, \psi_1, \dots, \psi_N, \dots), \quad (14)$$

where  $\psi_N$  is an  $N$ -particle wave function, defined on  $\Sigma^N$ . Corresponding to the structure of Fock space, the configuration space is

$$\mathcal{Q} = \bigcup_{N=0}^{\infty} \mathcal{Q}_N, \quad (15)$$

where  $\mathcal{Q}_N$  is the configuration space of  $N$  particles, as defined in (12). The Fock state  $\psi$  can then be regarded as a function on (the covering space)

$$\widehat{\mathcal{Q}} = \bigcup_{N=0}^{\infty} \widehat{\mathcal{Q}}_N. \quad (16)$$

(Both  $\mathcal{Q}_0$  and  $\widehat{\mathcal{Q}}_0$  contain just one element, the empty configuration.)

Thus, in the theory we are developing, if one among  $N$  particles hits the singularity, the configuration jumps from  $\mathcal{Q}_N$  to  $\mathcal{Q}_{N-1}$  in a deterministic way:  $q \rightarrow f(q)$ , where the jump function  $f : \partial\mathcal{Q}_N \rightarrow \cup_{n < N} \mathcal{Q}_n$  is given by

$$f(q) = q \setminus \partial\Sigma. \quad (17)$$

Here,  $\partial\Sigma$ , the boundary of  $\Sigma$ , is the intersection of  $\Sigma$  with the singularity  $S = \partial\mathcal{M}$ ;  $\partial\mathcal{Q}_N$  is the set of configurations in which at least one particle is located on  $\partial\Sigma$ ; and (17) means that  $f$  just removes all particles located on  $\partial\Sigma$ , i.e., all particles that have arrived at the singularity.

The situation in which the configuration space is a manifold with boundaries has been considered before in [12]; we recapitulate the Bohmian dynamics of the configuration developed there. It includes deterministic jumps that occur whenever the configuration reaches the boundary  $\partial\mathcal{Q}$  of configuration space. We desire three properties of the theory: reversibility, equivariance, and the Markov property; as pointed out in [12], these properties entail stochastic jumps with a certain rate.

This leads us to a stochastic process  $(Q_t)_{t \in \mathbb{R}}$  in  $\mathcal{Q}$  consisting of smooth motion interrupted by jumps from the boundary to the interior or vice versa. The jumps from  $\partial\mathcal{Q}$  to  $\mathcal{Q}^\circ$  are deterministic and occur whenever the process hits the boundary. The jumps from  $\mathcal{Q}^\circ$  to  $\partial\mathcal{Q}$  are stochastic, and their rates are fully determined by requiring that (i) the process is Markovian and equivariant, and (ii) the construction is invariant under time reversal, in that the processes associated with suitably time-reversed wave functions are reverse to each other, in distribution.

The deterministic jump law, abstractly

$$Q_{\tau+} = f(Q_{\tau-}) \quad (18)$$

for a fixed mapping  $f : \partial\mathcal{Q} \rightarrow \mathcal{Q}^\circ$ , is in our case given by (17). Since we want the theory to be reversible, we must also allow for spontaneous jumps from interior points to boundary points. Since we want the process to be an equivariant Markov process, the rate for a jump from  $q' \in \mathcal{Q}^\circ$  to a surface element  $dq \subseteq \partial\mathcal{Q}$  must be, as one can derive,

$$\sigma_t(q' \rightarrow dq) = \frac{\tilde{j}^r(q, t)^+}{\tilde{j}^t(q', t)} \nu(dq, q'), \quad (19)$$

where  $s^+ = \max(s, 0)$  denotes the positive part of  $s \in \mathbb{R}$ ,  $\tilde{j}^\mu$  is the probability current relative to coordinate volume (as discussed in detail in Section 5.1 below),  $\tilde{j}^t$  is its time component and thus the probability density relative to coordinate volume,  $\tilde{j}^r$  is its component orthogonal to  $\partial\mathcal{Q}$  (in our case, the radial component), and  $\nu(dq, q')$  is the measure-valued function defined in terms of the volume measure  $\mu$  on  $\mathcal{Q}$  and the surface area measure  $\lambda$  on  $\partial\mathcal{Q}$  by

$$\nu(B, q) = \frac{\lambda(B \cap f^{-1}(dq))}{\mu(dq)}, \quad (20)$$

for every  $B \subseteq \partial\mathcal{Q}$ . In our case, the measure  $\mu$  is on  $\mathcal{Q}_N$  locally the product of  $N$  copies of the coordinate volume measure on  $\Sigma = [0, \infty) \times \mathbb{S}^2$ , and the measure  $\lambda$  is on  $\partial\mathcal{Q}_N$  locally the product of the surface area measure on  $\mathbb{S}^2$  and  $N - 1$  copies of the coordinate volume measure on  $\Sigma = [0, \infty) \times \mathbb{S}^2$ . Note that the set of configurations with two or more particles on the singularity has  $\lambda$ -measure zero. It follows that  $\nu(\cdot, q)$  is a measure concentrated on the set of those configurations that consist of  $q$  plus one further particle on the singularity  $\partial\Sigma \cong \mathbb{S}^2$ , and the measure is essentially a copy of the surface area

measure on  $\mathbb{S}^2$ . Thus, concretely, the creation rate density relative to the area measure on  $\mathbb{S}^2$  is

$$\sigma_t(q \rightarrow q \cup \omega) = \frac{\tilde{j}^r(q \cup \omega, t)^+}{\tilde{j}^t(q, t)}, \quad (21)$$

where we have simply written  $\omega$  for the point  $(r = 0, \omega)$  on the singularity  $S \cap \Sigma$ .

Note that a jump to a boundary point  $q$  at which the current is pointing *towards* the boundary,  $\tilde{j}^r(q, t) < 0$ , would not allow any continuation of the process since there is no trajectory starting from  $q$ . The problem is absent if the velocity at  $q$  is pointing *away* from the boundary,  $\tilde{j}^r(q, t) > 0$ . (We are leaving out the case  $\tilde{j}^r(q, t) = 0$ .) On the other hand, jumps from  $q$  to  $f(q)$  cannot occur when  $\tilde{j}(q, t)$  is pointing away from the boundary since in that case there is no trajectory arriving at  $q$ . Thus, the jumps must be such that at each time  $t$ , one of the transitions  $q \rightarrow f(q)$  or  $f(q) \rightarrow q$  is forbidden, and the decision is made by the sign of  $\tilde{j}^r(q, t)$ .

Given the law of motion

$$\frac{dQ_t^\mu}{dt} = \frac{\tilde{j}^\mu}{\tilde{j}^t} \quad (22)$$

together with the deterministic jump law (18) and stochastic jumps with rate (19), we obtain the following probability transport equation at  $q' \in \mathcal{Q}$ :

$$\frac{\partial \tilde{j}^t}{\partial t}(q', t) = -\partial_i \tilde{j}^i(q', t) - \sigma_t(\partial \mathcal{Q}, q') \tilde{j}^t(q', t) + \int_{\partial \mathcal{Q}} \nu(dq, q') \tilde{j}^r(q, t)^-. \quad (23)$$

For equivariance we need that (23), has the structure of the transport equation for  $\tilde{j}^t$  that follows from the Schrödinger equation. This is what our model achieves by means of a suitably modified Dirac evolution and a boundary condition at the singularity.

## 5 Concrete Model

### 5.1 Probability Current

When we speak of probability density, or probability current density, it plays a role that densities can be considered either relative to *coordinate volume* or relative to *invariant volume* (proper volume), and this leads us to considering two current vector fields on  $\mathcal{M}$  (and afterwards on  $\mathcal{Q}$ ), which we denote respectively  $\tilde{j}^\mu$  and  $j^\mu$ . We elucidate this below in more detail, beginning with the single-particle case. The reason we consider densities relative to coordinate volume is that they stay bounded at the singularity, whereas the relevant densities relative to invariant volume diverge.

The current given by a one-particle wave function,  $j^\mu = \bar{\psi} \gamma^\mu \psi$ , is the current relative to invariant volume, which means the following. For every spacelike hypersurface  $\Sigma \subseteq \mathcal{M}$ , let  $n^\mu(x, \Sigma)$  be the future-pointing unit normal vector on  $\Sigma$  at  $x \in \Sigma$ ; let  $\lambda_\Sigma(d^3x)$  be the 3-volume measure associated with the Riemannian 3-metric on  $\Sigma$  inherited from  $g_{\mu\nu}$ . The Bohmian trajectory associated with  $\psi$  is a randomly chosen integral curve  $L$

of  $j^\mu$  whose probability distribution  $\mathbb{P}$  is such that, for every spacelike hypersurface  $\Sigma$ , the trajectory's unique intersection point with  $\Sigma$  has distribution  $j^\mu n_\mu \lambda_\Sigma$ , i.e.,

$$\mathbb{P}(L \cap \Sigma \subseteq A) = \int_A j^\mu(x) g_{\mu\nu}(x) n^\nu(x, \Sigma) \lambda_\Sigma(d^3x), \quad (24)$$

where  $A$  is any subset of  $\Sigma$ ,  $L \cap \Sigma$  the (set containing only the) intersection point, and  $\mathbb{P}(L \cap \Sigma \subseteq A)$  the probability of this point lying in  $A$ .

When computing densities relative to coordinate volume, we regard the coordinate space, in our example  $\mathbb{R} \times [0, \infty) \times \mathbb{S}^2$ , as a Riemannian 4-manifold  $\tilde{\mathcal{M}}$  with metric  $\tilde{g}_{\mu\nu}$ , in our example

$$d\tilde{s}^2 = dt^2 + dr^2 + d\vartheta^2 + \sin^2 \vartheta d\varphi^2. \quad (25)$$

Every spacelike hypersurface  $\Sigma \subseteq \mathcal{M}$  can also be regarded as a hypersurface in  $\tilde{\mathcal{M}}$ ; let  $\tilde{n}^\mu(x, \Sigma)$  be the vector at  $x \in \Sigma$  that is normal in coordinates (i.e., in  $\tilde{g}_{\mu\nu}$ ) to  $\Sigma$ , has unit length in coordinates, and points to the same side of  $\Sigma$  as  $n^\mu(x, \Sigma)$ . Likewise, let  $\tilde{\lambda}_\Sigma(d^3x)$  be the 3-volume measure defined by the 3-metric inherited from  $\tilde{g}_{\mu\nu}$ . As we show presently, there is a vector field  $\tilde{j}^\mu$  such that, for every spacelike hypersurface  $\Sigma$ ,

$$\tilde{j}^\mu(x) \tilde{g}_{\mu\nu} \tilde{n}^\nu(x, \Sigma) \tilde{\lambda}_\Sigma(d^3x) = j^\mu(x) g_{\mu\nu}(x) n^\nu(x, \Sigma) \lambda_\Sigma(d^3x). \quad (26)$$

Indeed,

$$\tilde{j}^\mu(x) = \frac{d\varepsilon}{d\tilde{\varepsilon}}(x) j^\mu(x), \quad (27)$$

where  $\varepsilon(d^4x)$  is the invariant 4-volume measure associated with  $g_{\mu\nu}$  and  $\tilde{\varepsilon}(d^4x)$  the coordinate 4-volume measure (associated with  $\tilde{g}_{\mu\nu}$ ). Note that, as a consequence of the proportionality,  $\tilde{j}^\mu$  and  $j^\mu$  have the same integral curves (up to an irrelevant reparameterization).

To show (26) and (27), we use differential forms (see, e.g., [18]). Let  $\mathbf{j}$  be the 3-form obtained by inserting  $j^\mu$  into  $\varepsilon$ , the invariant volume 4-form associated with  $g_{\mu\nu}$ . Then [25] the distribution (24) can be written as

$$\mathbb{P}(L \cap \Sigma \subseteq A) = \int_A \mathbf{j} \quad (28)$$

for every subset  $A$  of  $\Sigma$ . That is,  $\mathbf{j}$  encodes the probability current without reference to any metric. Making the same step backwards with  $\tilde{g}_{\mu\nu}$ , we obtain that

$$\tilde{j}^\mu \tilde{\varepsilon}_{\mu\nu\kappa\lambda} = \mathbf{j}_{\nu\kappa\lambda} = j^\mu \varepsilon_{\mu\nu\kappa\lambda}, \quad (29)$$

which implies (27).

One can compute the Radon–Nikodym derivative between the two measures by

$$\frac{d\varepsilon}{d\tilde{\varepsilon}} = \frac{\sqrt{-\det g_{\mu\nu}}}{\sqrt{-\det \tilde{g}_{\mu\nu}}}. \quad (30)$$

In our example case,  $d\varepsilon/d\tilde{\varepsilon} = r^2$ . (Note that taking  $(0, \infty) \times \mathbb{S}^2$  as coordinate space differs by a factor of  $r^2$  from taking  $\mathbb{R}^3 \setminus \{0\}$ .) We set

$$\tilde{\gamma}^\mu(x) = \frac{d\varepsilon}{d\tilde{\varepsilon}}(x) \gamma^\mu(x), \quad (31)$$

so that  $\tilde{j}^\mu = \bar{\psi} \tilde{\gamma}^\mu \psi$ .

If  $\Sigma$  is not spacelike, the distribution of the intersection point  $L \cap \Sigma$  is still given by

$$\mathbb{P}(L \cap \Sigma \subseteq A) = \int_A \tilde{j}^\mu(x) \tilde{g}_{\mu\nu}(x) \tilde{n}^\nu(x, \Sigma) \lambda_\Sigma(d^3x) \quad (32)$$

for any  $A \subseteq \Sigma$ , provided that  $L$  intersects  $\Sigma$  at most once, and that  $\tilde{n}^\nu$  always points to the same side of  $\Sigma$  as  $\tilde{j}^\mu$ . In particular, if  $\Sigma = S$  is the singularity then we obtain that the probability distribution of the random point where  $L$  hits the singularity has density (in coordinates) equal to (the absolute value of)  $\tilde{j}^r$ , the radial component of  $\tilde{j}^\mu$ , except when  $\tilde{j}^r > 0$  so that there is a current *away from* the singularity.

Since we only need that  $\tilde{j}^\mu$  remains bounded at the singularity, we can allow  $j^\mu$  to diverge like  $r^{-2}$ . This means, since  $j^\mu = \bar{\psi} \gamma^\mu \psi$ , that  $\psi$  can diverge like  $r^{-1}$  while  $\gamma^\mu$  stays bounded, or  $\gamma^\mu$  can diverge like  $r^{-2}$  while  $\psi$  stays bounded. We choose the second option, so that relevant wave functions  $\psi$  remain bounded at the singularity. This is connected to the definition of the spin spaces at the singularity.

## 5.2 Spin Spaces at the Singularity

The spin spaces  $\mathcal{S}_x$  are standardly defined at points  $x \in \mathcal{M}^\circ$ , and we also need to define them at points  $x \in S$  in order to be able to talk of  $\psi(x)$  and  $\tilde{j}^\mu(x)$  for  $x \in S$ . As a preparation, let us look at how the tangent spaces  $T_x \mathcal{M}$  are defined for  $x \in S$ . (We find it useful to regard  $T_x \mathcal{M}$  as a vector space rather than half-space.) Using coordinates, we can simply say, for  $x = (t, r = 0, \omega)$ , that

$$T_x \mathcal{M} = \mathbb{R} \times \mathbb{R} \times T_\omega \mathbb{S}^2, \quad (33)$$

where the first factor  $\mathbb{R}$  is understood as the tangent space to the manifold  $\mathbb{R}$  at  $t$ , and the second as the tangent space to the manifold  $\mathbb{R}$  at 0.

As a part of this specification of tangent spaces, we intend a certain topology (and differentiable structure) on the bundle  $T\mathcal{M} = \cup_{x \in \mathcal{M}} T_x \mathcal{M}$ : for example, if we set  $x = (t, r, \omega), y = (t, 0, \omega)$ , keep  $t$  and  $\omega \in \mathbb{S}^2$  fixed and let  $r \rightarrow 0$  so that  $x \rightarrow y$ , then the unit normal to the level surface  $\Sigma_t$  of the  $t$  function,  $n^\mu(x)$ , converges to 0, as its coordinate components are  $(\lambda(r)^{-1/2}, 0, 0)$ , while the unit normal relative to  $\tilde{g}_{\mu\nu}$ ,  $\tilde{n}^\mu(x)$ , converges to  $\tilde{n}^\mu(y)$ . Thus, another way of defining the tangent spaces on the singularity consists in specifying a tetrad field in  $\mathcal{M}^\circ$  (i.e., a basis for every tangent space) that possesses a smooth extension to  $\partial\mathcal{M}$ . In our example, we can take as the tetrads in  $\mathcal{M}^\circ$  simply the coordinate bases (or, orthonormal bases relative to  $\tilde{g}_{\mu\nu}$ ).

In the same way, we define the spin space  $\mathcal{S}_x$  for  $x \in \partial\mathcal{M}$  by specifying a basis  $\tilde{b}_x$  of  $\mathcal{S}_x$  for every  $x \in \mathcal{M}^\circ$  and postulating that the field  $\tilde{b}_x$  possesses a smooth

extension to  $\partial\mathcal{M}$ . To this end, fix any  $x \in \mathcal{M}^\circ$  and begin with the coordinate basis  $\tilde{e}_x$  of  $T_x\mathcal{M}$ ,  $\tilde{e}_x = (\partial_t, \partial_r, \partial_\vartheta, \partial_\varphi)$ . By normalizing these four vectors, we transform them into an orthonormal basis (i.e., Lorentz frame, a.k.a. Minkowski tetrad)  $e_x = (\lambda^{-1/2}\partial_t, \lambda^{1/2}\partial_r, r^{-1}\partial_\vartheta, (r \sin \vartheta)^{-1}\partial_\varphi)$ . To this orthonormal basis there corresponds a basis<sup>1</sup>  $b_x$  of  $\mathcal{S}_x$ ; the correspondence is canonical up to an overall sign which we choose continuously in  $x$ ;  $b_x$  is an orthonormal basis relative to the scalar product  $\bar{\phi}\gamma^\mu(x)g_{\mu\nu}(x)n^\nu(x)\psi$ . This basis we rescale by  $\lambda(r)^{-1/4}(d\varepsilon/d\tilde{\varepsilon})^{-1/2}$  to obtain  $\tilde{b}_x$ . This completes the definition of  $\tilde{b}_x$ .

In other words, if we define the sesquilinear mappings  $\alpha, \tilde{\alpha} : \mathcal{S}_x \times \mathcal{S}_x \rightarrow \mathbb{C}T_x\mathcal{M}$  by

$$\alpha(\phi, \psi) = \bar{\phi}\gamma^\mu\psi, \quad \tilde{\alpha}(\phi, \psi) = \bar{\phi}\tilde{\gamma}^\mu\psi, \quad (34)$$

then the coefficients of  $\alpha$  relative to  $b_x$  and  $e_x$  are the usual three Dirac  $\alpha$  matrices together with  $\alpha^0 = I$ , the identity matrix, while the coefficients of  $\tilde{\alpha}$  relative to  $\tilde{b}_x$  and  $\tilde{e}_x$  are as follows:

$$(\tilde{\alpha}^t, \tilde{\alpha}^r, \tilde{\alpha}^\vartheta, \tilde{\alpha}^\varphi) = \left( \frac{1}{\lambda}I, \alpha^1, \frac{1}{\lambda^{1/2}r}\alpha^2, \frac{1}{\lambda^{1/2}r \sin \vartheta}\alpha^3 \right), \quad (35)$$

where the  $\alpha^i$  on the right hand side denote the standard Dirac  $\alpha$  matrices. This equation could also be taken as the definition of  $\tilde{b}_x$ .

From (35) it follows that the coordinate formula for the probability current is given by

$$\tilde{j}^\mu = \psi^* \tilde{\alpha}^\mu \psi \quad (36)$$

where the components of  $\psi$  are taken relative to the basis  $\tilde{b}_x$ , and the  $\tilde{\alpha}^\mu$  matrices are given by the right hand side of (35). Note that these matrices possess the following limits

$$\text{as } r \rightarrow 0: \quad (\tilde{\alpha}^t, \tilde{\alpha}^r, \tilde{\alpha}^\vartheta, \tilde{\alpha}^\varphi) = \left( 0, \alpha^1, \frac{1}{e}\alpha^2, \frac{1}{e \sin \vartheta}\alpha^3 \right), \quad (37)$$

with  $e$  the charge parameter in (7). As a consequence, the probability current at the singularity always has vanishing time component, i.e., is tangent to  $\Sigma_t$ .

### 5.3 Boundary Condition

We impose a boundary condition at the singularity on the wave function. Let  $z$  be a point on  $S \cap \Sigma$  and  $x_1, \dots, x_{N-1} \in \Sigma \setminus S$ . A Dirichlet boundary condition [23]

$$\psi(x_1, \dots, x_{N-1}, z) = 0 \quad (38)$$

would lead to the (for our purposes) uninteresting behavior that no particle ever reaches the singularity. The boundary condition that we want should allow that there can be a probability current into the singularity but make sure that that current into the boundary of  $\mathcal{Q}_N$  corresponds to the gain in probability in  $\mathcal{Q}_{N-1}$ .

---

<sup>1</sup>The same can be done in two-spinor calculus, where this basis is called a “spin frame” [21].

To this end, we choose two functions  $\phi_{\pm} : \mathbb{S}^2 \rightarrow \mathbb{C}^4$  such that for every  $\omega \in \mathbb{S}^2$ ,  $\phi_{\pm}(\omega)$  is a normalized eigenvector of  $\alpha^1$  with eigenvalue  $\pm 1$  (recall that the Dirac  $\alpha$  matrices are Hermitian  $4 \times 4$  matrices with eigenvalues  $\pm 1$ ). Being eigenvectors to different eigenvalues, the two are orthogonal,  $\phi_+(\omega)^* \phi_-(\omega) = 0$ . As a consequence,  $\phi_{\pm}$  are two orthogonal vectors in  $L^2(\mathbb{S}^2, \mathbb{C}^4)$ , each with norm  $\sqrt{4\pi}$ . Thus, the  $\phi_{\pm}$  span a 2-dimensional subspace  $W \subseteq L^2(\mathbb{S}^2, \mathbb{C}^4)$ . In the following we will simply regard the singularity  $S \cap \Sigma$  as  $\mathbb{S}^2$  and write  $\omega$  for a point on the singularity.

We present the boundary condition as the conjunction of two boundary conditions, the first of which reads

$$\left[ \omega \mapsto \psi(q, \omega) \right] \in W \otimes (\mathbb{C}^4)^{\otimes N} \quad (39)$$

for every  $q \in \widehat{\mathcal{Q}}_N$ . Equivalently,  $\psi(q, \omega)$  can be written as

$$\psi(q, \omega) = \left( c_+(q) \phi_+(\omega) + c_-(q) \phi_-(\omega) \right) \otimes \chi(q) \quad (40)$$

with complex coefficients  $c_{\pm}(q)$  and suitable  $\chi(q) \in (\mathbb{C}^4)^{\otimes N}$ . The second boundary condition reads

$$\psi(q) = \frac{1}{\sqrt{8\pi}} (c_+(q) + c_-(q)) \chi(q), \quad (41)$$

where  $\chi(q)$  denotes the same spinor as in (40). Equivalently, we can write (41) as

$$\psi(q) = \frac{1}{\sqrt{8\pi}} (\phi_+(\omega) + \phi_-(\omega))^* \psi(q, \omega) \quad \forall \omega \in \mathbb{S}^2. \quad (42)$$

So we have specified what the boundary conditions are. They are obviously linear.

Technically speaking, the boundary conditions are part of the specification of the Hamiltonian  $H$ , as to specify  $H$  means to specify (i) its domain (a dense subspace of Hilbert space  $\mathcal{H}$ ) and (ii) how  $H$  acts on vectors in the domain. The domain consists of those wave functions from the first Sobolev space (i.e., possessing a square-integrable first derivative in the sense of distributions) that fulfill the boundary conditions described above. On such a function  $H$  acts as follows: For  $q \in \mathcal{Q}^\circ$ ,

$$H\psi(q) = -i\hbar \tilde{\alpha}^t(q)^{-1} \tilde{\alpha}^i(q) \partial_i \psi(q) + V(q) \psi(q) + H_I \psi(q), \quad (43)$$

where the index  $i$  runs through  $3N$  dimensions if  $q \in \widehat{\mathcal{Q}}_N$ , and  $V(q)$  assumes values in the Hermitian matrices on  $(\mathbb{C}^4)^{\otimes N}$  and includes the mass term  $\beta m$ , all connection coefficients that may arise from covariant derivatives, the external electromagnetic field (if desired), and all other potentials. The first two terms are just the (appropriate version of the) Dirac equation, while the third term creates a link between  $\psi_N$  and  $\psi_{N+1}$ :

$$H_I \psi(q) = i\hbar \sqrt{2\pi} \int_{\mathbb{S}^2} (\phi_+(\omega) - \phi_-(\omega))^* \psi(q, \omega) d\omega = \quad (44)$$

$$= i\hbar \sqrt{32\pi^3} (c_+(q) - c_-(q)) \chi(q). \quad (45)$$

What needs to be checked is the conservation of probability. At  $\{q \cup \omega : \omega \in \mathbb{S}^2\}$ , the probability current into the singularity is

$$\int_{\mathbb{S}^2} \tilde{j}^r(q \cup \omega) d\omega = \int_{\mathbb{S}^2} \psi(q, \omega)^* \tilde{\alpha}^r(q, \omega) \psi(q, \omega) d\omega = \quad (46)$$

$$= \int_{\mathbb{S}^2} \psi(q, \omega)^* \tilde{\alpha}^t(q) \otimes \alpha^1 \psi(q, \omega) d\omega = \quad (47)$$

$$= \frac{\chi(q)^* \chi(q)}{\prod_{x \in q} \lambda(r(x))} \int_{\mathbb{S}^2} \left( c_+(q) \phi_+(\omega) + c_-(q) \phi_-(\omega) \right)^* \alpha^1 \left( c_+(q) \phi_+(\omega) + c_-(q) \phi_-(\omega) \right) d\omega = \quad (48)$$

$$= \frac{\chi(q)^* \chi(q)}{\prod_{x \in q} \lambda(r(x))} \int_{\mathbb{S}^2} \left( c_+(q) \phi_+(\omega) + c_-(q) \phi_-(\omega) \right)^* \left( c_+(q) \phi_+(\omega) - c_-(q) \phi_-(\omega) \right) d\omega = \quad (49)$$

$$= 4\pi \frac{\chi(q)^* \chi(q)}{\prod_{x \in q} \lambda(r(x))} \left( |c_+(q)|^2 - |c_-(q)|^2 \right). \quad (50)$$

On the other hand, since the Dirac equation implies the conservation of probability, the gain of probability at  $q \in \mathcal{Q}^\circ$  is the contribution to the increase of  $\tilde{j}^t(q)$  due to  $H_I$ :

$$\left. \frac{\partial \tilde{j}^t}{\partial t} \right|_{H_I} = \frac{2}{\hbar} \text{Im} \psi(q)^* \tilde{\alpha}^t H_I \psi(q) = \frac{2}{\hbar \prod_{x \in q} \lambda(r(x))} \text{Im} \psi(q)^* H_I \psi(q) = \quad (51)$$

$$= 4\pi \frac{\chi(q)^* \chi(q)}{\prod_{x \in q} \lambda(r(x))} \text{Re} \left( c_+(q) + c_-(q) \right)^* \left( c_+(q) - c_-(q) \right) = \quad (52)$$

$$= 4\pi \frac{\chi(q)^* \chi(q)}{\prod_{x \in q} \lambda(r(x))} \left( |c_+(q)|^2 - |c_-(q)|^2 \right), \quad (53)$$

which is the same as (50). This shows that probability is conserved. We thus believe that (43), together with the boundary conditions, defines a self-adjoint Hamiltonian and thus a unitary evolution on Fock space.

## 6 Comparison to Bell-Type Quantum Field Theories

Two ways of extending Bohmian mechanics to quantum field theory are known: either by postulating that a field configuration (rather than a particle configuration) is guided by a wave function (understood as a functional on the field configuration space) [4, 5, 24], or by introducing particle creation and annihilation into Bohmian mechanics [2, 9, 10, 12] (but see also [6] for a third proposal). The second approach is called ‘‘Bell-type quantum



field theory,” as the first model of this kind (on a lattice) was proposed by Bell [2]. In these theories, the motion of the configuration along deterministic trajectories is interrupted by stochastic jumps, usually corresponding to the creation or annihilation of particles. The jumps are governed by the following law prescribing the jump rate  $\sigma^\psi$  (probability per time) in terms of the wave function  $\psi$ , which is usually from Fock space:

$$\sigma^\psi(q' \rightarrow q) = \frac{2}{\hbar} \frac{[\text{Im} \langle \psi | q \rangle \langle q | H_I | q' \rangle \langle q' | \psi \rangle]^+}{\langle \psi | q' \rangle \langle q' | \psi \rangle}, \quad (54)$$

where  $H_I$  is the interaction Hamiltonian. The configuration  $Q_t$  thus follows a Markov process in the configuration space  $\mathcal{Q}$  of a variable number of particles, as defined in (15) above; see [10] for a detailed discussion of this process.

We expect that further analysis will show that our model for stochastic particle creation at a timelike singularity fits into the scheme of Bell-type quantum field theories when understood in the appropriate way. In other words, we expect that the jump rate (21) can be regarded as a special case of (54).

## 7 Requirements on the Time Foliation

In the presence of singularities, difficulties arise about the choice of the time foliation that had not been considered previously. While the foliation we used on the Reissner–Nordström space-time, given by the  $t$  coordinate, worked well for our purposes, there also exist problematic foliations. Here is an example: Start with the  $\{t = 0\}$  hypersurface and propagate it according to (6), i.e., push every point of the hypersurface to the future at the same rate (so that the infinitesimal slice between two nearby hypersurfaces has constant proper thickness). The hypersurfaces we thus obtain are the level surfaces of the function  $T$  that yields, for a space-time point  $x$ , the timelike distance of  $x$  from the hypersurface  $\{t = 0\}$ . At first sight, this might seem like a reasonable foliation, but in fact it does not cover all of  $\mathcal{M}$ , not even  $\mathcal{M}^\circ$ . That is because the  $T$  function is infinite in a large region of space-time, in fact (in the future of  $\{t = 0\}$ ) at all points on or later than the outgoing radial null geodesics starting at  $t = 0$ . Let me explain.

I begin with describing the outgoing radial null geodesics of the Reissner–Nordström geometry: They are the integral curves of the vector field  $u^\mu = \lambda(r)^{-1/2} \partial_t + \lambda(r)^{1/2} \partial_r$ , as this vector is null (easy to check), radial ( $u^\vartheta = 0 = u^\varphi$ ), future-pointing ( $u^t > 0$ ), and outward-pointing ( $u^r > 0$ ). Thus, the outgoing radial null geodesics are explicitly given by

$$t(r) = t(0) + \int_0^r \frac{dr'}{\lambda(r')} \quad (55)$$

with constant  $\vartheta$  and  $\varphi$ . For small  $r$ ,  $t(r) \approx r^3/3e^2$ . Correspondingly, the *incoming* radial null geodesics are given by

$$t(r) = t(0) - \int_0^r \frac{dr'}{\lambda(r')}. \quad (56)$$

Note that at every point of the singularity, one incoming radial null geodesic ends and one outgoing radial null geodesic begins.

Consider the curve (55) for any  $t(0) > 0$  and call it  $C$ . Any point  $x$  on  $C$  must have  $T = \infty$ : If it had finite  $T$  value, then choose  $r_1$  so small that  $\lambda(r_1) > T(x)/t(0)$  and  $r_1 < r(x)$ ; that is possible because  $\lambda(r) \rightarrow \infty$  as  $r \rightarrow 0$ . Let  $x_1 = (t(r_1), r_1, \vartheta(x), \varphi(x))$ , which lies on  $C$ , too. Then the parallel to the  $t$  axis (in coordinate space) through  $x_1$  is a timelike curve connecting  $x_1$  to the hypersurface  $\{t = 0\}$ , and the proper time along this curve between  $x_1$  and  $\{t = 0\}$  is  $\lambda(r_1)t(r_1) > \lambda(r_1)t(0) > T(x)$ . Since  $T(x_1)$  is the supremum of such curve lengths, we have that  $T(x_1) > T(x)$ , so that  $x$  must lie in the past of the spacelike hypersurface  $\{T = T(x_1)\}$ , in contradiction to the fact that  $x$  lies on the future light cone of  $x_1$ .

Since  $t(0) > 0$  was arbitrary, it follows that any point in the future of the hypersurface defined by (55) for  $t(0) = 0$  and arbitrary  $\vartheta, \varphi$ , must have  $T = \infty$ . And thus,  $T$  foliates only a part of  $\mathcal{M}^\circ$ . Moreover, the leaves cannot be extended in a disjoint way to the singularity, as any  $\{T = \text{const.}\}$  hypersurface has  $\{(t = 0, r = 0)\} \times \mathbb{S}^2$  as its boundary on the singularity. One could say that the leaves intersect on the singularity.

In contrast, the  $t$  function provides a foliation of all of  $\mathcal{M}$ , including the singular boundary. That is a property we need of the time foliation.

*Acknowledgments.* This research was supported by grant RFP1-06-27 from The Foundational Questions Institute (fqxi.org).

## References

- [1] Baez, J.C., and Muniain, J.P.: *Gauge Fields, Knots, and Gravity*. Singapore: World Scientific (1994)
- [2] Bell, J. S.: Beables for Quantum Field Theory. *Phys. Rep.* **137**, 49–54 (1986). Reprinted as chapter 19 of [3].
- [3] Bell, J. S.: *Speakable and Unsayable in Quantum Mechanics*. Cambridge: Cambridge University Press (1987)
- [4] Bohm, D.: A Suggested Interpretation of the Quantum Theory in Terms of “Hidden” Variables, I and II. *Physical Review* **85**: 166–193 (1952)
- [5] Bohm, D., and Hiley, B.J.: *The Undivided Universe: An Ontological Interpretation of Quantum Theory*. London and New York: Routledge (1993)
- [6] Colin, S., and Struyve, W.: A Dirac sea pilot-wave model for quantum field theory. *J. Phys. A: Math. Theor.* **40**: 7309–7341 (2007). arXiv:quant-ph/0701085
- [7] Dürr, D., Goldstein, S., Münch-Berndl, K., and Zanghì, N.: Hypersurface Bohm–Dirac Models. *Phys. Rev. A* **60**: 2729–2736 (1999). arXiv:quant-ph/9801070
- [8] Dürr, D., Goldstein, S., Tumulka, R., and Zanghì, N.: Trajectories and Particle Creation and Annihilation in Quantum Field Theory. *J. Phys. A: Math. Gen.* **36**: 4143–4149 (2003). arXiv:quant-ph/0208072

- [9] Dürr, D., Goldstein, S., Tumulka, R., and Zanghì, N.: Bohmian Mechanics and Quantum Field Theory. *Phys. Rev. Lett.* **93**: 090402 (2004). arXiv:quant-ph/0303156
- [10] Dürr, D., Goldstein, S., Tumulka, R., and Zanghì, N.: Bell-Type Quantum Field Theories. *J. Phys. A: Math. Gen.* **38**: R1–R43 (2005). arXiv:quant-ph/0407116
- [11] Dürr, D., Goldstein, S., and Zanghì, N.: Quantum Equilibrium and the Origin of Absolute Uncertainty. *J. Statist. Phys.* **67**: 843–907 (1992). arXiv:quant-ph/0308039
- [12] Georgii, H.-O., and Tumulka, R.: Some Jump Processes in Quantum Field Theory. In J.-D. Deuschel, A. Greven (eds), *Interacting Stochastic Systems*, pp. 55–73. Berlin: Springer-Verlag (2005). arXiv:math.PR/0312326
- [13] Goldstein, S. (2001): Bohmian Mechanics. In E. N. Zalta (ed.), *Stanford Encyclopedia of Philosophy*, published online by Stanford University. <http://plato.stanford.edu/entries/qm-bohm/>
- [14] Goldstein, S., and Tumulka, R.: Opposite Arrows of Time Can Reconcile Relativity and Nonlocality. *Class. Quantum Gravity* **20**: 557–564 (2003). arXiv:quant-ph/0105040
- [15] Hawking, S. W., and Ellis, G. F. R.: *The large scale structure of space-time*. Cambridge: Cambridge University Press (1973)
- [16] Lang, S.: *Differentiable Manifolds*. Reading, Mass.: Addison Wesley (1972)
- [17] Maudlin, T.: *Quantum Non-Locality and Relativity: Metaphysical Intimations of Modern Physics*. Oxford: Basil Blackwell (1994)
- [18] Misner, C.W., Thorne, K.S., and Wheeler, J.A.: *Gravitation*. New York: Freeman (1973)
- [19] Nordström, G.: Zur Theorie der Gravitation vom Standpunkt des Relativitätsprinzips. *Ann. Phys. (Germany)* **42**: 533–554 (1913)
- [20] Nordström, G.: On the energy of the gravitational field in Einstein’s theory. *Proc. Kon. Ned. Akad. Wet.* **20**: 1238–1245 (1918)
- [21] Penrose, R., and Rindler, W.: *Spinors and space-time. Volume 1: Two-spinor calculus and relativistic fields*. Cambridge: Cambridge University Press (1984)
- [22] Reissner, H.: Über die Eigengravitation des elektrischen Feldes nach der Einsteinschen Theorie. *Ann. Phys.* **50**: 106–120 (1916)
- [23] Stalker, J. G., and Tahvildar-Zadeh, A. S.: Scalar waves on a naked-singularity background. *Class. Quantum Gravity* **21**: 2831–2848 (2004)

- [24] Struyve, W.: Field beables for quantum field theory. arXiv:0707.3685
- [25] Tumulka, R.: *Closed 3-Forms and Random Worldlines*. Ph. D. thesis, Mathematics Institute, Ludwig-Maximilians-Universität, München, Germany (2001)
- [26] Tumulka, R.: The ‘unromantic pictures’ of quantum theory. *J. Phys. A: Math. Theor.* **40**: 3245–3273 (2007). arXiv:quant-ph/0607124
- [27] Weyl, H.: Gravitation und Elektrizität. *Sitzungs-Berichte der Preußischen Akademie der Wissenschaften* 465 (1918)